Course Information

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Course Contents:
1. Review of Quantum Mechanics  
2. Field Quantization  
3. Coherent and Squeezed States  
4. Quantum Theory of Atom-field Interaction  
5. Jaynes-Cummings Model  
6. Quantum Coherence Theory  
7. Quantum Damping Theory  
8. Cavity QED

Pre-requirements:
Quantum Mechanics and Classical Electrodynamics

Reference Books:

Grades:
The homework weights 60% and an oral presentation on a journal paper weights 40%.
Homework No. 1
Due on 20th October 2010

   3. Give a proof of the Baker–Hausdorff lemma: for any two operators $\hat{A}$ and $\hat{B}$,
      \[ e^{i\hat{A}} \hat{B} e^{-i\hat{A}} = \hat{B} + i\lambda [\hat{A}, \hat{B}] + \frac{(i\lambda)^2}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \cdots \]

   4. In the special case where $[\hat{A}, \hat{B}] \neq 0$, but where $[\hat{A}, [\hat{A}, \hat{B}]] = 0 = [\hat{B}, [\hat{A}, \hat{B}]]$, show that
      \[ e^{i\hat{A}+\hat{B}} = \exp \left( -\frac{1}{2} [\hat{A}, \hat{B}] \right) e^\hat{A} e^\hat{B} = \exp \left( \frac{1}{2} [\hat{A}, \hat{B}] \right) e^\hat{B} e^\hat{A}. \]
      This is known as the Baker–Hausdorff–Campbell theorem.

   6. Consider the superposition state $|\psi_{01}\rangle = \alpha |0\rangle + \beta |1\rangle$ where $\alpha$ and $\beta$ are complex and satisfy $|\alpha|^2 + |\beta|^2 = 1$. Calculate the variances of the quadrature operators $\hat{X}_1$ and $\hat{X}_2$. Are there any values of the parameters $\alpha$ and $\beta$ for which either of the quadrature variances become less than for a vacuum state? If so, check to see if the uncertainty principle is violated. Repeat with the state $|\psi_{02}\rangle = \alpha |0\rangle + \beta |2\rangle$.

   8. Consider the superposition of the vacuum and 10 photon number state
      \[ |\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |10\rangle). \]
      Calculate the average photon number for this state. Next assume that a single photon is absorbed and recalculate the average photon number. Does your result seem sensible in comparison with your answer to the previous question?

\[ k_x = \frac{2\pi n_x}{L}, \quad k_y = \frac{2\pi n_y}{L}, \quad k_z = \frac{2\pi n_z}{L}, \tag{1.1.21} \]

1.1 The radiation field in an empty cubic cavity of side \( L \) satisfies the wave equation

\[ \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = 0, \]

together with the Coulomb gauge condition \( \nabla \cdot A = 0 \). Show that the solution that satisfies the boundary conditions has components

\[ A_x(r, t) = A_x(t) \cos(k_x x) \sin(k_y y) \sin(k_z z), \]
\[ A_y(r, t) = A_y(t) \sin(k_x x) \cos(k_y y) \sin(k_z z), \]
\[ A_z(r, t) = A_z(t) \sin(k_x x) \sin(k_y y) \cos(k_z z), \]

where \( A(t) \) is independent of position and the wave vector \( \mathbf{k} \) has components given by Eq. (1.1.21). Hence show that the integers \( n_x, n_y, n_z \) in Eq. (1.1.21) are restricted in that only one of them can be zero at a time.


1.6 Show that the free-field Hamiltonian

\[ \mathcal{H} = \hbar \nu \left( a^\dagger a + \frac{1}{2} \right) \]

can be written in terms of the number states as

\[ \mathcal{H} = \sum_n E_n |n\rangle \langle n|, \]

and hence

\[ e^{i\mathcal{H}t/\hbar} = \sum_n e^{i E_n t/\hbar} |n\rangle \langle n|. \]
Homework No. 2  
Due on 10th November 2010


4. Prove the following identities:

\[
\hat{a}^\dagger |\alpha\rangle \langle \alpha| = \left( \alpha^* + \frac{\partial}{\partial \alpha} \right) |\alpha\rangle \langle \alpha|, \\
|\alpha\rangle \langle \alpha| \hat{a} = \left( \alpha + \frac{\partial}{\partial \alpha^*} \right) |\alpha\rangle \langle \alpha|.
\]


5. Verify Eq. (3.16), that the quantum fluctuations of the field quadrature operators are the same as for the vacuum when the field is in a coherent state.

\[
\star \quad \langle (\Delta \hat{X}_1)^2 \rangle_\alpha = \frac{1}{4} = \langle (\Delta \hat{X}_2)^2 \rangle_\alpha
\]

(3.16)


7. For a coherent state $|\alpha\rangle$ evaluate the sine and cosine operators of Eqs. (2.211) and their squares. (You will not get closed forms.) Examine the limit where the average photon number $\bar{n} = |\alpha|^2 \gg 1$. Then examine the uncertainty products of Eqs. (2.215) and (2.216) in this limit.

\[
\star \quad \hat{C} \equiv \frac{1}{2}(\hat{E} + \hat{E}^\dagger), \quad \hat{S} \equiv \frac{1}{2i}(\hat{E} - \hat{E}^\dagger)
\]

\[
(\Delta n)(\Delta C) \geq \frac{1}{2} |\langle \hat{S} \rangle| \quad (2.215)
\]

\[
(\Delta n)(\Delta S) \geq \frac{1}{2} |\langle \hat{C} \rangle| \quad (2.216)
\]

13. Consider the superposition state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\beta\rangle + |\beta\rangle)$$

where the $|\pm\beta\rangle$ are coherent states. (a) Show that this state is normalized for the case where $|\beta|^2 \gg 1$. (b) Obtain the photon number probability distribution. (c) Obtain the phase distribution. (d) Obtain the $Q$ and Wigner functions for this state and display them as three-dimensional plots. Is $|\psi\rangle$ a classical state?


2.3 The time evolution of the wave packet (2.3.14) is determined by the Schrödinger equation for the harmonic oscillator

$$i\hbar \frac{\partial \psi}{\partial t} = \left( -\frac{\hbar^2}{2} \frac{\partial^2}{\partial q^2} + \frac{v^2 q^2}{2} \right) \psi.$$  

A general solution of this equation can be given in terms of the stationary wave functions

$$\psi(q, t) = \sum_{n=0}^{\infty} a_n \phi_n(q) e^{-iE_n t / \hbar},$$

where $E_n = (n+1/2)\hbar v$ and $a_n$ are arbitrary coefficients. Using the orthonormality conditions on the wave functions $\phi_n(q)$, find $a_n$ and hence prove Eq. (2.3.15).

$$\star \psi(q, 0) = \left( \frac{v}{\pi \hbar} \right)^{1/4} \exp \left[ -\frac{v}{2\hbar} (q - q_0)^2 \right]. \quad (2.3.14)$$

$$|\psi(q, t)|^2 = \left( \frac{v}{\pi \hbar} \right)^{1/2} \exp \left[ -\frac{v}{\hbar} (q - q_0 \cos vt)^2 \right]. \quad (2.3.15)$$


2.5 An alternate definition of a squeezed coherent state is

$$|\alpha, \zeta\rangle = D(\alpha) S(\zeta)|0\rangle,$$

where $\zeta = r \exp(i\theta)$. Show that the variances in the quadrature components $Y_1$ and $Y_2$, such that

$$Y_1 + iY_2 = ae^{-i\theta/2},$$

are given by

$$(\Delta Y_1)^2 = \frac{1}{4} e^{-2r},$$

$$(\Delta Y_2)^2 = \frac{1}{4} e^{-2\theta}.$$

2.8 Consider the Hermitian operators corresponding to the real and imaginary parts of the square of the complex amplitude of the field

\[ X_1 = \frac{1}{2}(a^2 + a^\dagger^2), \]

\[ X_2 = \frac{1}{2i}(a^2 - a^\dagger^2). \]

Show that the squeezing condition is

\[ \langle \Delta X_i^2 \rangle < \langle a^\dagger a \rangle + \frac{1}{2} \quad (i = 1 \text{ or } 2). \]

This type of squeezing is called amplitude-squared squeezing. Show that the amplitude-squared squeezing is a nonclassical effect. (Hint: see M. Hillery, *Phys. Rev. A* 36, 3796 (1987).)

5. In the text, we obtained the dynamics of the Jaynes–Cummings model assuming exact resonance, $\Delta = 0$. Reconsider the problem for the case where $\Delta \neq 0$. Obtain plots of the atomic inversion and note the effect of the nonzero detuning on the collapse and revivals of the Rabi oscillations. Perform an analysis to obtain the effect of the nonzero detuning on the collapse and revival times.


7. Consider a simple model of degenerate Raman scattering, pictured in Fig. 4.12 (where $E_g = E_e$), and described by the interaction Hamiltonian $\hat{H}_I = \hbar \lambda \hat{a}^\dagger \hat{a} (\sigma_+ + \sigma_-)$, where, as usual, $\sigma_+ = |e\rangle \langle g|$ and $\sigma_- = |g\rangle \langle e|$.

(a) Obtain the dressed states for this model.

(b) Assuming initially the field in a coherent state and the atom in the ground state, obtain the atomic inversion and show that the revivals of the Rabi oscillations are regular and complete.

(c) Obtain the atomic inversion for an initial thermal state.

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Fig. 4.12. Energy-level diagram for the degenerate Raman coupled model. The broken line represents a “virtual” intermediate state, too far off-resonance from a real level, the upper solid line, to become populated.

8. A resonant two-photon extension of the Jaynes–Cummings model is described by the effective Hamiltonian $H_{\text{eff}} = \hbar \eta (\hat{a}^2 \hat{\sigma}_+ + \hat{a}^{2\dagger} \hat{\sigma}_-)$, where, for the sake of simplicity, a small Stark shift term has been ignored. This Hamiltonian represents two-photon absorption and emission between atomic levels of like parity. The process is represented by Fig. 4.13, where the broken line represents a virtual intermediate state of opposite parity.

(a) Obtain the dressed states for this system.

(b) Obtain the atomic inversion for this model assuming the atom initially in the ground state and that the field is initially in a number state. Repeat for a coherent state. Comment on the nature of the collapse and revival phenomena for these states.

(c) Obtain the atomic inversion for an initial thermal state.

![Fig. 4.13. Energy-level diagram for the resonant two-photon process. States $|e\rangle$ and $|g\rangle$ are of like parity whereas the intermediate state $|i\rangle$ is of opposite parity. The broken line represents a virtual atomic level, detuned from state $|i\rangle$.](image)


9. A two-mode variation on the two-photon model of the previous problem is described by the Hamiltonian $H_{\text{eff}} = \hbar \eta (\hat{a} \hat{b} \hat{\sigma}_+ + \hat{a}^{\dagger} \hat{b}^{\dagger} \hat{\sigma}_-)$, that is, a photon from each mode is absorbed or emitted. Obtain the atomic inversion for the case where both modes are initially in coherent states. Analyse the collapse and revival phenomena.

5.2 The finite lifetime of the atomic levels can be described by adding phenomenological decay terms to the probability amplitude equations (5.2.12) and (5.2.13):

$$
\dot{c}_a = -\frac{\gamma}{2} c_a + \frac{\Omega_R}{2} e^{-i\phi} c_b,
$$

$$
\dot{c}_b = -\frac{\gamma}{2} c_b + \frac{\Omega_R}{2} e^{i\phi} c_a,
$$

where $\gamma$ is the decay constant and $\omega = \nu$. For an atom initially in the state $|a\rangle$, show that the inversion at time $t$ is

$$
W(t) = e^{-it\gamma} \cos(\Omega_R t).
$$


5.5 Consider a three-level atom interacting with a classical field of frequency $\nu$. The transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$ are allowed whereas the transition $|a\rangle \rightarrow |c\rangle$ is forbidden. It is also assumed that $\omega_a - \omega_b = \omega_b - \omega_c = \nu$. Assuming the atom to be initially in level $|c\rangle$, find the probabilities for the atom to be in levels $|a\rangle$ and $|c\rangle$ after making the rotating-wave approximation.


6.1 A model sometimes considered to study the atom-field coupling in a lossless cavity is represented by the Hamiltonian

$$
\mathcal{H} = \hbar \nu a^+ a + \hbar \omega \sigma_z + \hbar g \left[ \sigma_+ a (a^+ a)^{1/2} + (a^+ a)^{1/2} a^+ \sigma_- \right],
$$

in the usual notation. Note that the coupling is intensity dependent. Calculate the atomic inversion and discuss its evolution in terms of the various time scales, i.e., Rabi flopping time, the collapse time, and the revival time, for (a) an initial coherent state of the field and (b) an initial thermal state of the field. Note that the infinite series in the expression for inversion can be summed exactly in this case.

4. Consider the superposition state of the vacuum and one-photon states,

\[ |\psi\rangle = C_0 |0\rangle + C_1 |1\rangle, \]

where \( |C_0|^2 + |C_1|^2 = 1 \), and investigate its coherence properties. Note that quantum mechanically it is “coherent” because it is a pure state. Compare your result with that obtained for the mixture

\[ \rho = |C_0|^2 |0\rangle \langle 0| + |C_1|^2 |1\rangle \langle 1|. \]


4.3 Consider a state described by the density operator

\[ \rho = \mathcal{N} e^{-\kappa a^* a}, \]

where \( \mathcal{N} \) is a normalization constant and \( \kappa = \hbar \nu / k_B T \).

(a) Show that it goes over to a Fock state if \( \kappa \to \infty \) and to a thermal state if \( \kappa \to 0 \).

(b) Find \( g^{(2)}(0) \) and show that the photon statistics are sub-Poissonian if

\[ \bar{n} < \sqrt{\frac{m}{m + 1}}, \]

where \( \bar{n} = [\exp(\kappa) - 1]^{-1} \).

**8.2** The equation of motion for the reduced density operator for a single-mode cavity field coupled to a vacuum reservoir through a partially transmitting mirror is

\[ \dot{\rho} = -\frac{\mathcal{C}}{2} (a^\dagger a \rho - 2 a \rho a^\dagger + \rho a^\dagger a). \]

Here \( \mathcal{C} \) is the loss rate related to the \( Q \)-factor of the cavity by \( \mathcal{C} = \nu/Q \). Derive the equations of motion for the relevant quantities, and then solve them to show that the variances \( (\Delta X_1)_t^2 \) and \( (\Delta X_2)_t^2 \) (with \( X_1 = (a + a^\dagger)/2 \) and \( X_2 = (a - a^\dagger)/2i \)) increase due to dissipation (fluctuation–dissipation theorem!). This situation can be viewed as a bosonic mode, uncorrelated to the cavity field, entering the cavity through the partially transmitting mirror, and hence adding the uncorrelated noise.


**8.3** If the reservoir in the above problem is in a multi-mode squeezed vacuum state, the resulting equation of motion for the reduced density matrix is given by Eq. (8.3.4). As before, calculate the variances \( (\Delta X_1)_t^2 \) and \( (\Delta X_2)_t^2 \). Is it possible to suppress the added noise in this situation?

\[
\dot{\rho} = -\frac{\mathcal{C}}{2} (N + 1)(a^\dagger a \rho - 2 a \rho a^\dagger + \rho a^\dagger a)
\]
\[
- \frac{\mathcal{C}}{2} N(aa^\dagger \rho - 2a^\dagger \rho a + \rho aa^\dagger)
\]
\[
+ \frac{\mathcal{C}}{2} M(aa \rho - 2a \rho a + \rho aa)
\]
\[
+ \frac{\mathcal{C}}{2} M^*(a^\dagger a^\dagger \rho - 2a^\dagger \rho a^\dagger + \rho a^\dagger a^\dagger), \tag{8.3.4}
\]

9.1 A single mode of frequency $\nu$ interacts with a thermal reservoir. The evolution of the field–reservoir system is described by the Langevin equation

$$\dot{\hat{a}} = -\frac{1}{2} \mathcal{E} \hat{a} + F_a(t),$$

where $\hat{a}(t) = a(t)e^{i\nu t}$, $a$ is the destruction operator for the field mode. Calculate the variance $(\Delta X_1)^2$ (with $X_1 = (\hat{a} + \hat{a}^\dagger)/2$) at a time $t$ in terms of the variance at the initial time $t = 0$.


9.2 Find the correlation function $\langle F_a^\dagger(t)F_a(t') \rangle$ in Eq. (9.2.13) for

$$f(t_i, t, \tau) = \begin{cases} e^{-\Gamma(t - t_i)} & \text{for } t_i \leq t < t_i + \tau, \\ 0 & \text{otherwise.} \end{cases}$$

$$\langle F_a^\dagger(t)F_a(t') \rangle = g^2 \sum_{ij} f(t_i, t, \tau)f(t_j, t', \tau)\langle \sigma_+^i(t_i)\sigma_-^j(t_j) \rangle$$

$$= g^2 [1 + \exp(h\nu/k_B T)]^{-1} \sum_i f(t_i, t, \tau)f(t_i, t', \tau).$$

(9.2.13)